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Variance Calculations and the Bessel Kernel

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In the Laguerre ensemble of $N \times N$ Hermitian matrices, it is of interest both theoretically and for applications to quantum transport problems to compute the variance of a linear statistic, denoted $\mathrm{var}_N f$, as $N \to \infty$. Furthermore, this statistic often contains an additional parameter α for which the limit $\alpha \to \infty$ is most interesting and most difficult to compute numerically. We derive exact expressions for both $\lim_{N \to \infty} \mathrm{var}_N f$ and $\lim_{\alpha \to \infty} \lim_{N \to \infty} \mathrm{var}_N f$.

KEY WORDS: Random matrices; Laguerre ensemble; variance; quantum transport.

1. INTRODUCTION AND STATEMENT OF MAIN RESULT

In the random matrix theory of quantum transport (see refs. 1, 2, and 6 and references therein) the following quantity is fundamental[3]:

$$\mathrm{var}_N f := \int_0^\infty f^2\left(\frac{(4N\mu)^{1/2}}{\alpha}\right) K_N(\mu, \mu)\, d\mu - \int_0^\infty \int_0^\infty f\left(\frac{(4N\mu)^{1/2}}{\alpha}\right) f\left(\frac{(4N\mu')^{1/2}}{\alpha}\right) K_N^2(\mu, \mu')\, d\mu\, d\mu' \quad (1.1)$$

where $K_N(\mu, \mu')$ is the Laguerre kernel; that is,

$$K_N(\mu, \mu') = \sum_{j=0}^{N-1} \phi_j(\mu)\, \phi_j(\mu')$$

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[3] The notational choice $f(\sqrt{x})$ rather than $f(x)$ in (1.1) will be convenient later. It also agrees with the convention of Stone *et al.*[6]

and $\{\phi_j(x)\}$ is the sequence of functions obtained by orthonormalizing the sequence

$$\{x^j x^{\nu/2} e^{-x/2}\}_{j=0}^{\infty}$$

over $(0, \infty)$ (here $\nu > -1$). In particular, one is interested in

$$\operatorname{var} f := \lim_{N \to \infty} \operatorname{var}_N f \tag{1.2}$$

in the limit $\alpha \to \infty$. In applications various choices are made for f, but we need assume here only that f is smooth and sufficiently decreasing at infinity to make the integrals well-defined.

In the random matrix model of disordered conductors, the quantity $\operatorname{var} f$ is related (via the two-probe Landauer formula) to the fluctuations of the conductance and the limit $\alpha \to \infty$ is the high-density (or metallic) regime. A lucid account can be found in the review article by Stone *et al.*,[6] to which we refer the reader for further details and references. However, these authors did not evaluate $\operatorname{var}_N f$ in the limits of interest; namely $N \to \infty$ followed by $\alpha \to \infty$. It is the purpose of this paper to evaluate these limits. We will see that the result agrees with the prediction of Beenakker,[1,2] who gave a heuristic argument for this limit.

By a change of variables we write (1.1) in the more suggestive form

$$\begin{aligned} \operatorname{var}_N f = & \int_0^\infty f^2\left(\frac{\sqrt{x}}{\alpha}\right) \frac{1}{4N} K_N\left(\frac{x}{4N}, \frac{y}{4N}\right) dx \\ & - \int_0^\infty \int_0^\infty f\left(\frac{\sqrt{x}}{\alpha}\right) f\left(\frac{\sqrt{y}}{\alpha}\right) \\ & \times \left(\frac{1}{4N} K_N\left(\frac{x}{4N}, \frac{y}{4N}\right)\right)^2 dx\, dy \end{aligned} \tag{1.3}$$

From asymptotic formulas for generalized Laguerre polynomials (see, e.g., 10.15.2 in ref. 3) it follows that[4,7]

$$\begin{aligned} K(x, y) &:= \lim_{N \to \infty} \frac{1}{4N} K_N\left(\frac{x}{4N}, \frac{y}{4N}\right) \\ &= \frac{J_\nu(\sqrt{x})\sqrt{y}\, J_\nu'(\sqrt{y}) - \sqrt{x}\, J_\nu'(\sqrt{x})\, J_\nu(\sqrt{y})}{2(x-y)} \end{aligned} \tag{1.4}$$

where $J_\nu(z)$ is the Bessel function of order ν. (The limit is uniform in x and y for $0 < x, y \leqslant L < \infty$ and all L.) We call $K(x, y)$ the "Bessel kernel." (This kernel also arises in scaling the Jacobi ensemble of random matrices at either edge ± 1.) Using this in (1.3), we obtain

$$\operatorname{var} f = \int_0^\infty f^2\left(\frac{\sqrt{x}}{\alpha}\right) K(x, x)\, dx - \int_0^\infty \int_0^\infty f\left(\frac{\sqrt{x}}{\alpha}\right) f\left(\frac{\sqrt{y}}{\alpha}\right) K^2(x, y)\, dx\, dy \tag{1.5}$$

where K is the Bessel kernel.

The problem is reduced to evaluating (1.5) in the limit $\alpha \to \infty$. We will show that

$$\lim_{\alpha \to \infty} \operatorname{var} f = \frac{1}{\pi^2} \int_{-\infty}^\infty |\hat{f}(2iy)|^2\, y \tanh(\pi y)\, dy \tag{1.6}$$

where $\hat{f}$ is the Mellin transform of f, i.e.,

$$\hat{f}(z) = \int_0^\infty x^{z-1} f(x)\, dx$$

This agrees with the result of Beenakker[1,2] once one notes that his $f(x)$ is our $f(\sqrt{x})$. For numerous applications of (1.6) we refer the reader to Beenakker.[2]

2. THE LIMIT $\alpha \to \infty$

2.1. Use of Hankel Transform

It is convenient to define the kernel

$$\begin{aligned} L(x, y) &:= 2K(x^2, y^2) \\ &= \int_0^1 t J_\nu(xt)\, J_\nu(yt)\, dt \end{aligned} \tag{2.1}$$

where $K(x, y)$ is the Bessel kernel. (A simple proof of the second equality can be found in ref. 7.) Then $\operatorname{var} f$ can be written

$$\operatorname{var} f = \int_0^\infty x f^2\left(\frac{x}{\alpha}\right) L(x, x)\, dx - I_1 \tag{2.2}$$

where

$$
\begin{aligned}
I_1 &= \int_0^\infty \int_0^\infty xyf\left(\frac{x}{\alpha}\right) f\left(\frac{y}{\alpha}\right) L^2(x, y)\, dx\, dy \\
&= \int_0^1 \int_0^\infty \left(\int_0^1 \left(\int_0^\infty xf\left(\frac{x}{\alpha}\right) J_\nu(xt)\, J_\nu(xt')\, dx \right) t' J_\nu(t'y)\, dt' \right) \\
&\quad \times yf\left(\frac{y}{\alpha}\right) tJ_\nu(ty)\, dy\, dt \\
&= \int_0^1 \int_0^\infty \left(\int_0^\alpha \left(\int_0^\infty xf(x)\, J_\nu(t\alpha x)\, J_\nu(t'x)\, dx \right) t' J_\nu\left(\frac{t'y}{\alpha}\right) dt' \right) \\
&\quad \times yf\left(\frac{y}{\alpha}\right) tJ_\nu(ty)\, dy\, dt
\end{aligned} \tag{2.3}
$$

where we used (2.1) to deduce the middle equality and we made the change of variables $x/\alpha \to x$ and $\alpha t' \to t'$ to obtain the last equality.

We now recall the Hankel inversion formula:

$$
\int_0^\infty u \left(\int_0^\infty xg(x)\, J_\nu(xu)\, dx \right) J_\nu(u\xi)\, du = g(\xi)
$$

which holds for $\sqrt{x}\, g(x)$ continuous and absolutely integrable on the positive real line and $\nu > -1/2$. First writing the t'-integration in (2.3) as the integral from $(0, \infty)$ minus the integral from (α, ∞) and then employing the Hankel inversion formula [with the choice $g(x) = f(x)\, J_\nu(\alpha t x)$] on the part containing the t'-integration from $(0, \infty)$, we see that this part *exactly* cancels the single integral appearing in the expression (2.2) for $\operatorname{var} f$. Thus we are left with

$$
\begin{aligned}
\operatorname{var} f &= \int_0^1 dt \int_0^\infty dy \int_\alpha^\infty dt' \int_0^\infty dx \\
&\quad \times xytt' f(x)\, f\left(\frac{y}{\alpha}\right) J_\nu(\alpha t x)\, J_\nu\left(\frac{t'y}{\alpha}\right) J_\nu(t'x)\, J_\nu(ty) \\
&= \alpha^4 \int_0^1 dt \int_0^\infty dy \int_1^\infty ds \int_0^\infty dx \\
&\quad \times xyts f(x)\, f(y)\, J_\nu(\alpha t x)\, J_\nu(\alpha s x)\, J_\nu(\alpha s y)\, J_\nu(\alpha t y)
\end{aligned} \tag{2.4}
$$

We remark that the Hankel transform plays the analogous role for the Bessel kernel that the Fourier transform plays for the sine kernel

$$\frac{1}{\pi}\frac{\sin \pi(x-y)}{x-y}$$

in the Gaussian unitary ensemble.

2.2. Residue Calculation

Introducing the (inverse) Mellin transform

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{f}(z)\, x^{-z}\, dz \qquad (c>0)$$

into (2.4) and interchanging the orders of integration, we see that

$$\begin{aligned}\operatorname{var} f = \alpha^4 \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz_1\, \hat{f}(z_1)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz_2\, \hat{f}(z_2)\\ \times \int_0^1 dt\, t\int_1^\infty ds\, s\int_0^\infty dx\, x^{-z_1+1} J_\nu(\alpha t x)\, J_\nu(\alpha s x)\\ \times \int_0^\infty dy\, y^{-z_2+1} J_\nu(\alpha t y)\, J_\nu(\alpha s y)\end{aligned}$$

The x and y integrations can be performed using (6.5762) in ref. 5; namely,

$$\begin{aligned}\int_0^\infty x^{-\lambda} J_\nu(ax)\, J_\nu(bx)\, dx = \frac{(ab)^\nu\, \Gamma(\nu+(1-\lambda)/2)}{2^\lambda (a+b)^{2\nu-\lambda+1}\, \Gamma(1+\nu)\, \Gamma((1+\lambda)/2)}\\ \times F\left(\nu+\frac{1-\lambda}{2}, \nu+\frac{1}{2}; 2\nu+1; \frac{4ab}{(a+b)^2}\right)\end{aligned}$$

where $F(a, b; c; z)$ is the hypergeometric function, $a, b>0$, $2\Re(\nu)+1 > \Re(\lambda) > -1$.

In the resulting integral we make the following change of variables:

$$u=\frac{4ts}{(t+s)^2}, \qquad v=t+s$$

which has Jacobian

$$J(u, v)=\frac{v}{4(1-u)^{1/2}}$$

In the vu plane we are now integrating over the region in the first quadrant bounded above by the curve

$$u=\frac{4(v-1)}{v^2}, \qquad v \geqslant 1$$

and bounded below by the ray $[1, \infty]$ on the v axis. The v integration may now be trivially done with the result that

$$\begin{aligned}\operatorname{var} f = {} & \frac{1}{4^{2\nu-1}\Gamma^2(\nu+1)} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz_1\, \hat{f}(z_1)\, \alpha^{z_1} \frac{\Gamma(\nu+1-z_1/2)}{\Gamma(z_1/2)} \\ & \times \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz_2\, \hat{f}(z_2)\, \alpha^{z_2} \frac{\Gamma(\nu+1-z_2/2)}{\Gamma(z_2/2)} \\ & \times \frac{1}{z_1+z_2}\int_0^1 u^{2\nu+1-z_1-z_2}(1-u)^{-1/2} \\ & \times [(1+(1-u)^{1/2})^{z_1+z_2}-(1-(1-u)^{1/2})^{z_1+z_2}] \\ & \times F_\nu(z_1, u)\, F_\nu(z_2, u)\, du\end{aligned}$$

where

$$F_\nu(z, u) := F\left(\nu+1-\frac{z}{2}, \nu+\frac{1}{2}; 2\nu+1; u\right)$$

We now use z_1 and $z=z_1+z_2$ as integration variables so that

$$\begin{aligned}\operatorname{var} f = {} & \frac{1}{4^{2\nu-1}\Gamma^2(\nu+1)} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz_1\, \hat{f}(z_1) \frac{\Gamma(\nu+1-z_1/2)}{\Gamma(z_1/2)} \\ & \times \frac{1}{2\pi i}\int_{2c-i\infty}^{2c+i\infty} dz\, \hat{f}(z-z_1)\, \alpha^z \frac{\Gamma(\nu+1-(z-z_1)/2)}{\Gamma((z-z_1)/2)} \\ & \times \int_0^1 u^{2\nu+1-z}(1-u)^{-1/2}\left[\frac{(1+(1-u)^{1/2})^z-(1-(1-u)^{1/2})^z}{z}\right] \\ & \times F_\nu(z_1, u)\, F_\nu(z-z_1, u)\, du\end{aligned}$$

Observe that the α dependence of $\operatorname{var} f$ resides solely in the term α^z in the above integral. To compute $\alpha \to \infty$ this suggests we should first deform the contour into the left-half z plane. The $\lim_{\alpha\to\infty} \operatorname{var} f$ will be determined by the residue of the pole at $z=0$.

To calculate this residue (which is a function of z_1) we must know the principal part of the Laurent expansion (in z) of the integral involving the u integration. The divergence of this integral as $z \to 0$ is determined by the behavior of the integrand in the vicinity of $u=1$. This behavior near $u=1$ is straightforward to compute since it is known[3] that

$$F(a, b; c, u) \sim \frac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)}(1-u)^{c-a-b} \qquad \text{as} \quad u \to 1$$

Thus $\lim_{\alpha\to\infty} \operatorname{var} f$ is expressed as a single integral over the variable z_1. If we now make use of the Γ-function identities[3]

$$\Gamma(z)\,\Gamma(-z) = -\frac{\pi}{z \sin \pi z}$$

$$\Gamma(1/2+z)\,\Gamma(1/2-z) = \frac{\pi}{\cos \pi z}$$

$$\Gamma(2\nu+1) = 2^{2\nu}\pi^{-1/2}\Gamma(\nu+1/2)\,\Gamma(\nu+1)$$

we obtain

$$\lim_{\alpha\to\infty} \operatorname{var} f = -\frac{1}{2\pi^2 i}\int_{c-i\infty}^{c+i\infty} \hat{f}(z_1)\,\hat{f}(-z_1)\,\frac{z_1}{2}\tan\left(\frac{\pi}{2}z_1\right) dz_1$$

We now deform the contour to the imaginary axis (and send $y \to 2y$) to obtain (1.6).

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